Nonlinear dynamics of ultrasound waves radiated from cavitation bubbles

Sören Schenke[1], Fabian Sewerin[1,2], Berend van Wachem[1], Fabian Denner[1,3]

soeren.schenke@ovgu.de
mvt.ovgu.de

[1]Chair of Mechanical Process Engineering
[2]Emmy Noether Group for Dispersed Multiphase Flows
[3]Center for Advanced Medical Engineering

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

Workshop Kavitation
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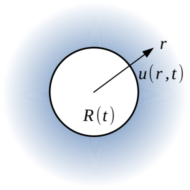
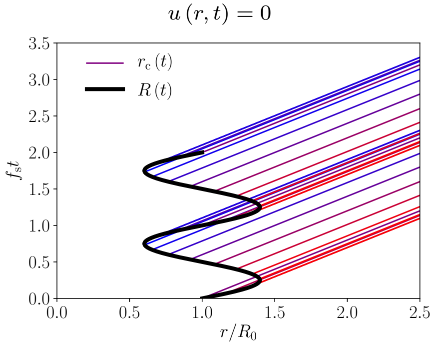
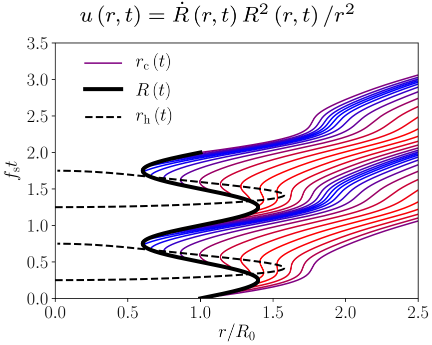
Acoustic waves in a transonic flow

Goal

- Solving the wave equation along the characteristic
$$r_c(t) = R(t - t_0) + \int_{t_0}^{t} (c_0 + u(r_c, t))\, dt.$$
- Accounting for thermo-viscous attenuation.







Waves from moving boundaries in a moving medium

Westervelt equation

$$\frac{\partial^2 p}{\partial t^2} \underbrace{- \frac{\delta}{c_0^2}\frac{\partial^3 p}{\partial t^3}}_{\text{sound diffusion}} \underbrace{- \frac{\beta}{\rho_0 c_0^2}\frac{\partial^2}{\partial t^2}\left(p^2\right)}_{\text{nonlinear distortion}} = c_0^2 \frac{\partial^2 p}{\partial x^2}$$

P. J. Westervelt, JASA **35**, 535 (1963).

Convective spherical form:
$$\frac{\mathrm{D}^2 p}{\mathrm{D}t^2} - \frac{\delta}{c_0^2}\frac{\mathrm{D}^3 p}{\mathrm{D}t^3} - \frac{\beta}{\rho_0 c_0^2}\frac{\mathrm{D}^2}{\mathrm{D}t^2}\left(p^2\right) = c_0^2 \underbrace{\left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial x}\right)}_{\text{spherical Laplacian}}$$

Material derivative: $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ → motion of the fluid

Moving physical domain: $\Omega(t) = [R_\mathrm{l}(t), R_\mathrm{r}(t)]$ → motion of the boundary

Fixed computational domain: $\Theta = [\mathcal{X}_\mathrm{l}, \mathcal{X}_\mathrm{r}]$

Coordinate transformation: $r : \Theta \to \Omega(t), \ (\xi, t) \mapsto r(\xi, t)$

Coordinate transformation on the physical space $\Omega(t)$

Convective Westervelt equation:

$$\frac{\mathrm{D}^2 p}{\mathrm{D}t^2} - \frac{\delta}{c_0^2}\frac{\mathrm{D}^3 p}{\mathrm{D}t^3} - \frac{\beta}{\rho_0 c_0^2}\frac{\mathrm{D}^2}{\mathrm{D}t^2}\left(p^2\right) = c_0^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial x}\right).$$

Example of the material derivative:

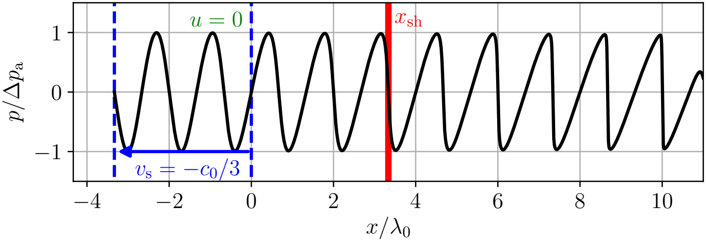
$$\frac{\mathrm{D}^3 p}{\mathrm{D}t^3} = \frac{\partial^3 p}{\partial t^3} + \left(2u\frac{\partial^2 u}{\partial r \partial t} + \frac{\mathrm{D}u}{\mathrm{D}t}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial t^2} + u^2\frac{\partial^2 u}{\partial r^2}\right)\frac{\partial p}{\partial r} + 3u\frac{\mathrm{D}u}{\mathrm{D}t}\frac{\partial^2 p}{\partial r^2} + u^3\frac{\partial^3 p}{\partial r^3} + 3\frac{\mathrm{D}u}{\mathrm{D}t}\frac{\partial^2 p}{\partial r \partial t} + 3u^2\frac{\partial^3 p}{\partial r^2 \partial t} + 3u\frac{\partial^3 p}{\partial r \partial t^2}.$$

Example of the coordinate transformation on $\Omega(t)$:

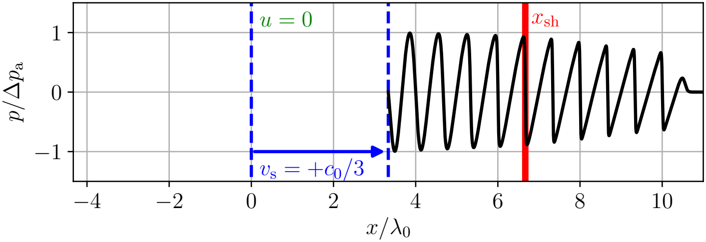
$$\frac{\partial^3 p}{\partial r^2 \partial t} = \left(\det \mathrm{J}_r(\xi)\right)^2\left(2\mathrm{div}_\xi(q)\frac{\partial^2 \mathcal{P}}{\partial \xi^2} + q\frac{\partial^3 \mathcal{P}}{\partial \xi^3} + \frac{\partial^3 \mathcal{P}}{\partial \xi^2 \partial t}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\det \mathrm{J}_r(\xi)\right)\left(q\frac{\partial^2 \mathcal{P}}{\partial \xi^2} + \frac{\partial^2 \mathcal{P}}{\partial \xi \partial t}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\det \mathrm{J}_r(\xi)\,\mathrm{div}_\xi(q)\right)\frac{\partial \mathcal{P}}{\partial \xi}.$$

Doppler shift of the shock formation distance x_{sh}

Red shift:



Blue shift:

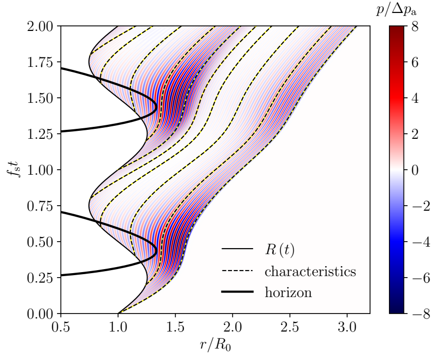


Sonic horizon around a pulsating sphere

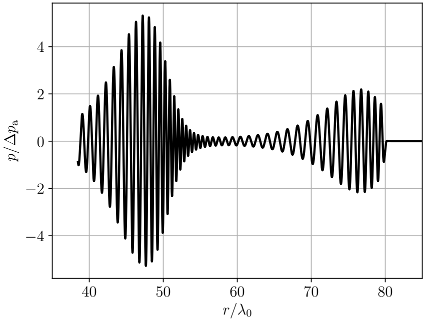
Mach number: $\mathrm{Ma} = \left|\dot{R}_{\max}\right| / c_0 = 1.6$

Helmholtz number: $\mathrm{He} = 2\pi R_0 / \lambda_0 = 201$

Acoustic pressure in the r-t plane



Wave profile at $f_s t = 1$



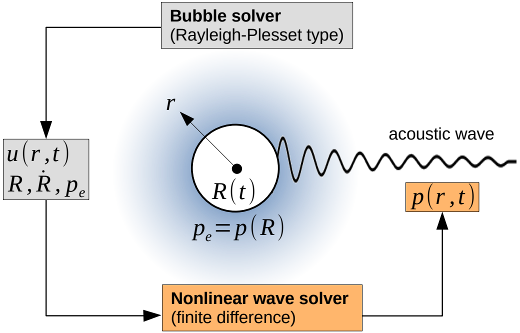
Observation: Nonlinear Doppler modulation of the wave amplitude.

Coupled bubble-wave solver

Gilmore equation

$$\left(1-\frac{\dot{R}}{c_\mathrm{L}}\right)R\ddot{R}+\frac{3}{2}\left(1-\frac{\dot{R}}{3c_\mathrm{L}}\right)\dot{R}^2=\left(1+\frac{\dot{R}}{c_\mathrm{L}}\right)H+\left(1-\frac{\dot{R}}{c_\mathrm{L}}\right)R\frac{\dot{H}}{c_\mathrm{L}}$$

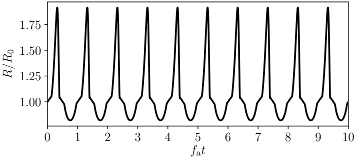
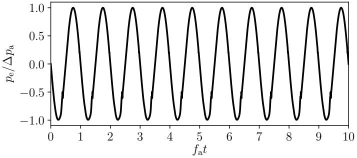
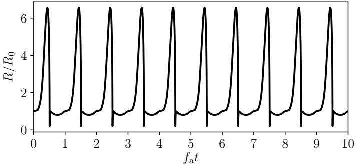
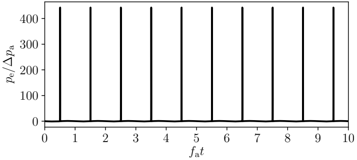
F. R. Gilmore, Technical Report No. 26-4, California Institute of Technology, 1952.



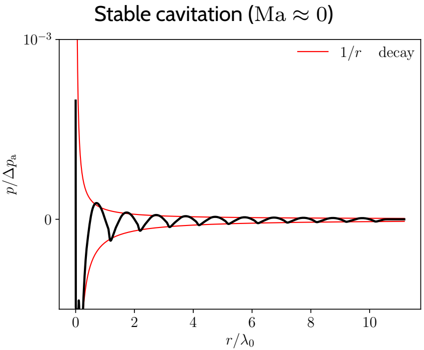
Testcase: Lipid coated SonoVue bubble

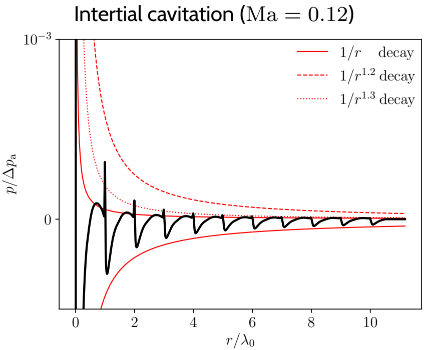
- $R_0 = 1\ \mu\text{m}$
- excitation frequency: $f_\text{a} = 100\ \text{kHz}$

J. Gümmer, S. Schenke, F. Denner, Ultrasound in Medicine and Biology, **47**, 10 (2021).

	Bubble radius over time	Bubble wall liquid pressure over time
Stable cavitation $\Delta p_\text{a} = 170\ \text{kPa}$ $\text{Ma} \approx 0$		
Inertial cavitation $\Delta p_\text{a} = 183\ \text{kPa}$ $\text{Ma} = 0.12$		

Spatial acoustic pressure wave profiles

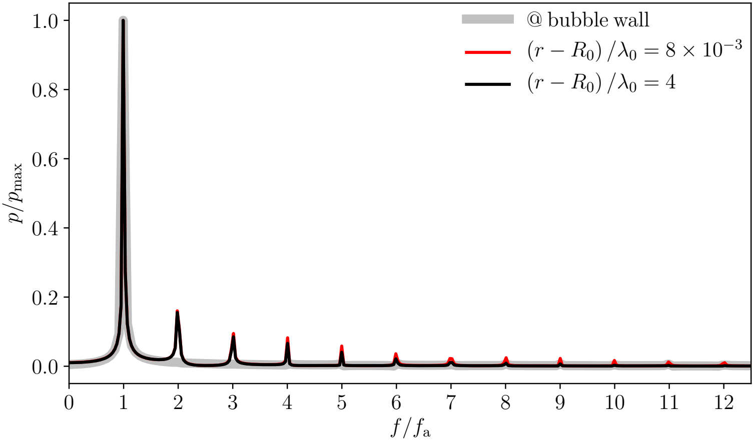




The sinusoidal excitation signal appears to be distorted.

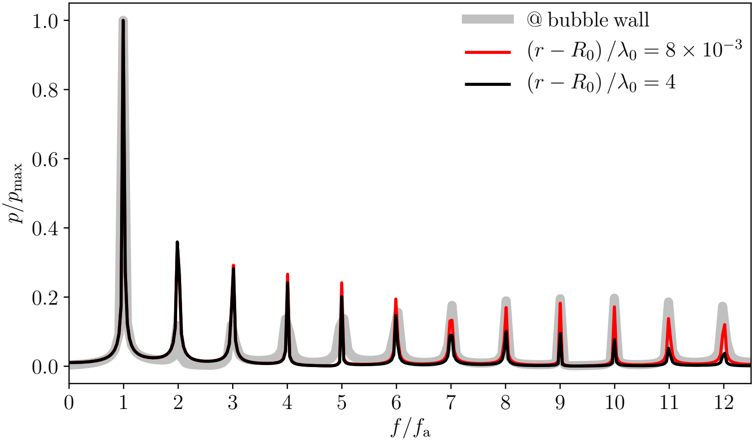
Peak pressures decay at a rate higher than $1/r$.

Pressure spectrum for stable cavitation ($\mathrm{Ma} \approx 0$)



The pressure radiation at a varying bubble radius causes a geometrical distortion of the pressure signal that reflects in higher harmonics.

Pressure spectrum for inertial cavitation ($\mathrm{Ma} = 0.1$)



The high wave harmonics associated with the collapse pressure pulses are strongly attenuated by thermo-viscous damping.

Conclusion & outlook

Observations

- We do not observe any Doppler effects around oscillating bubbles.
- Additional wave harmonics at some distance from the bubble wall.

Explanations

Helmholtz number: $\mathrm{He} = 2\pi \frac{R_0}{\lambda_0}.$

- The velocity field is only "noticeable" in very close proximity to the bubble wall.
- The change of bubble wall position has a geometrical effect on the outgoing wave profile.
- Geometrical and thermo-viscous damping strongly attenuates higher wave harmonics.

`contact: soeren.schenke@ovgu.de`

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Fast Lagrangian reconstruction of shock attenuation

Fay solution (saw-tooth envelope): $$\hat{p}_{\mathrm{Fay}}\left(x\right) = \frac{\pi \Delta p_{\mathrm{a}}}{1 + x/x_{\mathrm{sh}}}$$

R. D. Fay, JASA **3**(2A), 1931.

Local wave decay: $$p_{\mathsf{Fay}}\left(x,t\right) = \frac{\pi p_{\mathrm{e}}}{1 + x\beta p_{\mathrm{e}} / \left(\rho_0 c_0^3 \sqrt{-\dot{p}_{\mathrm{e}}/\dddot{p}_{\mathrm{e}}}\right)}$$

Shock formation: $$x\left(t+\delta t\right)\big|_{t_{\mathrm{e}}} = x\left(t\right)\big|_{t_{\mathrm{e}}} + \left(c_0 + \frac{p_{\mathsf{Fay}}}{\rho_0 c_0}\right)\delta t \leq x\left(t+\delta t\right)\big|_{t_{\mathrm{e}}-\delta t}$$